KMA315 Analysis 3A: Solutions to Problems 1

1. Find the infimum (greatest lower bound) and supremum (least upper bound) of the following subsets of $\mathbb{R}$ (justify your claims):

(i) $\left\{\frac{2}{n+1} : n \in \mathbb{N}\right\}$; *(4 marks)*

(ii) $\left\{\frac{(-1)^n}{n^3} : n \in \mathbb{Z}_+\right\}$; *(4 marks)*

(iii) $\left\{n^{(-1)^n} : n \in \mathbb{N}\right\}$. *(4 marks)*

(i) Let $A = \left\{\frac{2}{n+1} : n \in \mathbb{N}\right\} = \left\{2, \frac{2}{3}, \frac{2}{4}, \ldots\right\} \subseteq (0, 2]$. Since $0 < \frac{2}{n+1} \leq 2$ for all $n \in \mathbb{N}$, it follows that 0 is a lower bound of A and 2 is an upper bound of A.

Since $2 \in A$ *(when $n = 0$)*, A cannot have any upper bounds lower than 2, hence $\sup A = 2$.

Let $\varepsilon > 0$ and $N \in \mathbb{N}$. $\frac{2}{N+1} < \varepsilon$ is satisfied when $\frac{2}{\varepsilon} - 1 < N$. Hence for each $\varepsilon > 0$, $\frac{2}{n+1} < \varepsilon$ for all $n > \frac{\varepsilon}{2} - 1$, so $\inf A = 0$.

(ii) Let $B = \left\{\frac{(-1)^n}{n^3} : n \in \mathbb{Z}_+\right\} = \left\{-1, \frac{1}{8}, -\frac{1}{27}, \ldots\right\} \subseteq [-1, \frac{1}{8}]$. Since $-1 \leq \frac{(-1)^n}{n^2} \leq \frac{1}{8}$ for all $n \in \mathbb{N}$, it follows that -1 is a lower bound of B and $\frac{1}{8}$ is an upper bound of B.

Since $-1 \in B$ *(when $n = 1$)*, B cannot have any lower bounds greater than 2, hence $\inf B = -1$.

Since $\frac{1}{8} \in B$ *(when $n = 2$)*, B cannot have any upper bounds lower than $\frac{1}{8}$, hence $\sup A = \frac{1}{8}$.

(iii) Let $C = \left\{n^{(-1)^n} : n \in \mathbb{N}\right\} = \left\{0, 1, 2, \frac{1}{3}, 4, \frac{1}{5}, 6, \ldots\right\} \subseteq [0, \infty)$. Since $0 \leq n^{(-1)^n} < \infty$ for all $n \in \mathbb{N}$, it follows that 0 is a lower bound of C.

Since $0 \in C$ *(when $n = 0$)*, C cannot have any lower bounds greater than 0, hence $\inf C = 0$.

Note that $2\mathbb{Z}_{\geq 0} \subseteq C$ (ie. the non-negative even integers are a subset of C), hence for any $M \in \mathbb{R}$, $\{r \in 2\mathbb{Z}_{\geq 0} : r > M\}$ is non-empty and $\{r \in 2\mathbb{Z}_{\geq 0} : r > M\} \subseteq C$, so C does not have any upper bounds for there to be a least of.

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2. Let A be a subset of real numbers such that A contains only a finite number of elements. Is it possible for the greatest lower bound of A to <u>not</u> be an element of A? Justify your claim.
(2 marks)

A set containing only a finite number of elements trivially has a minimum element, which trivially by definition must be a lower bound. Any lower bound of a set that is also contained in the set must trivially be the greatest lower bound.

For a general subset $X \subseteq \mathbb{R}$, if $\min X$ exists and $\min X \in X$ then $\inf X = \min X$, which is trivially the case when $|X| < \infty$.

3. Determine and explain whether the following sequences are - (a) bounded above/below, (b) monotone increasing/decreasing, (c) convergent and (d) if they converge then also what their limit is:

(i) $((-1)^n)_{n=0}^{\infty} = (1, -1, 1, -1, \ldots)$; *(5 marks)*

(ii) $(\frac{2}{n+1})_{n=0}^{\infty} = (2, 1, \frac{2}{3}, \frac{1}{2}, \frac{2}{5}, \frac{1}{3}, \ldots)$. *(5 marks)*

(i) It is trivially the case that $-1 \leq (-1)^n \leq 1$ for all $n \in \mathbb{N}$. Hence $((-1)^n)_{n=0}^{\infty}$ is bounded below by -1 and bounded above by 1.

It is enough to note that within the first three terms $((-1)^n)_{n=0}^{\infty}$ decreases and then increases again to conclude that $((-1)^n)_{n=0}^{\infty}$ is not a monotonic sequence.

Since $((-1)^n)_{n=0}^{\infty}$ oscillates back and forth between 1 and -1, there is trivially no limit.

(ii) It is trivially the case that $0 < \frac{2}{n+1} \leq 2$ for all $n \in \mathbb{N}$. Hence $(\frac{2}{n+1})_{n=0}^{\infty}$ is bounded below by 0 and bounded above by 2.

It is trivially the case that $\frac{2}{n+1} < \frac{2}{n+2}$ for all $n \in \mathbb{N}$, and hence that $(\frac{2}{n+1})_{n=0}^{\infty}$ is monotonically decreasing.

Upon observing the first few terms, it appears as though $(\frac{2}{n+1})_{n=0}^{\infty}$ converges towards 0. Let $\varepsilon > 0$, in order for $N \in \mathbb{N}$ to satisfy $\frac{2}{N+1} < \varepsilon$, we require $\frac{2}{\varepsilon} - 1 < N$. It follows that for each $\varepsilon > 0$, $\frac{2}{n+1} < \varepsilon$ for all $n \geq \frac{2}{\varepsilon} - 1$, and hence $\lim_{n\to\infty} \frac{2}{n+1} = 0$.

4. Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be sequences of real numbers. Prove that:

(i) if $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are bounded above then $(a_nb_n)_{n=0}^{\infty}$ is also bounded above; *(3 marks)*

(ii) if $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are monotone decreasing then $(a_n+b_n)_{n=0}^{\infty}$ is also monotone decreasing; *(3 marks)*

(iii) if $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ converge then $(a_n+b_n)_{n=0}^{\infty}$ also converges
[also prove that in such a case we have $\lim_{n\to\infty}(a_n+b_n) = (\lim_{n\to\infty} a_n) + (\lim_{n\to\infty} b_n)$].
(3 marks)

(i) Let $(a_n)_{n=0}^{\infty} = (b_n)_{n=0}^{\infty} = (-n)_{n=0}^{\infty}$. Since $-n \leq 0$ for all $n \in \mathbb{N}$, both $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are bounded above. However $(a_nb_n)_{n=0}^{\infty} = (n^2)_{n=0}^{\infty}$ which is trivially <u>NOT</u> bounded above, hence it does not follow from $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ being sequences of real numbers that are bounded above that $(a_nb_n)_{n=0}^{\infty}$ will be a sequence of real numbers that is bounded above.

(ii) *Proof.* Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be sequences of monotonically decreasing real numbers. Hence $a_n < a_{n+1}$ and $b_n < b_{n+1}$ for all $n \in \mathbb{N}$. It trivially follows that $a_n+b_n < a_{n+1}+b_{n+1}$ for all $n \in \mathbb{N}$ and hence that $(a_b+b_n)_{n=0}^{\infty}$ is a monotonically decreasing sequence of real numbers. $\square$

(iii) *Proof.* Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be sequences of real numbers that converge to $L_a \in \mathbb{R}$ and $L_b \in \mathbb{R}$ respectively. Let $\varepsilon > 0$, since $\lim_{n\to\infty} a_n = L_a$ and $\lim_{n\to\infty} b_n = L_b$, there exists $N_a, N_b \in \mathbb{N}$ such that $|a_n - L_a| < \frac{\varepsilon}{2}$ for all $n \geq N_a$ and $|b_n - L_b| < \frac{\varepsilon}{2}$ for all $n \geq N_b$. It trivially follows that $|(a_n+b_n) - (L_a+L_b)| \leq |a_n - L_a| + |b_n - L_b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for all $n \geq \max\{N_a, N_b\}$, which establishes that $\lim_{n\to\infty}(a_n+b_n) = L_a + L_b$. Hence $\lim_{n\to\infty}(a_n+b_n) = (\lim_{n\to\infty} a_n) + (\lim_{n\to\infty} b_n)$. $\square$

5. Prove that if a sequence of real numbers is monotone decreasing and bounded below then it converges to its infimum (aka greatest lower bound). *(4 marks)*

Proof. Let $(y_n)_{n=0}^{\infty}$ be a sequence of real numbers that is monotone decreasing and bounded below. Since $\{y_n : n \in \mathbb{N}\}$ has a lower bound, it follows from the Greatest-Lower-Bound Property that $\{y_n : n \in \mathbb{N}\}$ has a greatest lower bound in the real numbers, which we shall denote by l (ie. $\inf \{y_n : n \in \mathbb{N}\} = l$).

Now, for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $y_N < l + \varepsilon$, since otherwise $l + \varepsilon$ would also be an upper bound, which would contradict l being the <u>greatest</u> lower bound. Furthermore, since $y_n \leq y_N$ for all $n > N$ (which follows from $(y_n)_{n=0}^{\infty}$ being monotone decreasing), we have $y_n < l + \varepsilon$ for all $n \geq N$. Rearranging we get $|y_n - l| < \varepsilon$ for all $n \geq N$, which establishes that $(y_n)_{n=0}^{\infty}$ converges to l. $\square$

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